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# BIPRODUCT BIALGEBRAS WITH A PROJECTION ONTO A HOPF ALGEBRA

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Abstract. Let $(D, B)$ be an admissible pair. Then recall that $B \times_H^L D \rightleftarrows_{i_D}^{\pi_D} D$ are bialgebra maps satisfying $\pi_D \circ i_D = I$. We have solved a converse in case $D$ is a Hopf algebra. Let $D$ be a Hopf algebra with antipode $s_D$ and be a left $H$-comodule algebra and a left $H$-module coalgebra over a field $k$. Let $A$ be a bialgebra over $k$. Suppose $A \rightleftarrows_i^\pi D$ are bialgebra maps satisfying $\pi \circ i = I_D$. Set $\Pi = I_D * (i \circ s_D \circ \pi), B = \Pi(A)$ and $j : B \to A$ be the inclusion. Suppose that $\Pi$ is an algebra map. We show that $(D, B)$ is an admissible pair and $B \leftrightarrows_j^\Pi A \rightleftarrows_i^\pi D$ is an admissible mapping system and that the generalized biproduct bialgebra $B \times_H^L D$ is isomorphic to $A$ as bialgebras.

Given algebras $A$ and $B$, we put an algebra structure on the tensor product $A \otimes B$ by

$$(a \otimes b)(a' \otimes b') = aa' \otimes bb' \cdots\cdots\cdots\cdots\cdots\cdots (0)$$

where $a,\ a' \in A$ and $b,\ b' \in B$. We call $A \otimes B$ the *tensor product of the algebras* $A$ *and* $B$. Its unit is $1 \otimes 1$. Defining $i_A(a) = a \otimes 1$ and $i_B(b) = 1 \otimes b$, we get algebra morphisms $i_A : A \longrightarrow A \otimes B$ and $i_B : B \longrightarrow A \otimes B$. The following relation holds in view of (0) :

$$i_A(a)i_B(b) = i_B(b)i_A(a) = a \otimes b$$

for all $a \in A$ and $b \in B$.

Molnar constructed a smash coproduct $C \sharp H$ of an $H$-comodule coalgebra $C$ and a Hopf algebra $H$ in [4] and usual smash product $A \# H$ of an $H$-module algebra $A$ and a Hopf algebra $H$ has been defined in [8] or [9].

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Definition 1 [1]. *Let $H$ be a bialgebra over a field $k$ and $A$ be a left $H$-module algebra. Let $D$ be a left $H$-comodule algebra. The generalized smash product* $A\#_H^L D$ is defined to be $A \otimes_k D$ as a vector space, with multiplication given by

$$(a\#_H^L d)(b\#_H^L e) = \Sigma a(d_{-1} \cdot b)\#_H^L d_0 e$$

and unit $1_A \otimes 1_D$ for all $a, b \in A$ and $d, e \in D$.

It is straightforward to show that $i_A : A \longrightarrow A\#_H^L D,\ a \longmapsto a\#_H^L 1_D$ and $i_D : D \longrightarrow A\#_H^L D,\ d \longmapsto 1_A\#_H^L d$ are algebra maps since $A$ is a left $H$-module algebra and $D$ is a left $H$-comodule algebra.

Definition 2 [2]. *Let $H$ be a bialgebra over a field $k$ and $C$ be a left $H$-comodule coalgebra. Let $E$ be a left $H$-module coalgebra. The generalized smash coproduct* $C\natural_H^L E$ is defined to be $C \otimes_k E$ as a vector space with comultiplication given by

$$\Delta(c\natural_H^L e) = \Sigma(c_1\natural_H^L c_{2,-1} \cdot e_1) \otimes (c_{2,0}\natural_H^L e_2)$$

and counit

$$\varepsilon(c\natural_H^L e) = \varepsilon_C(c)\varepsilon_E(e)$$

for all $c \in C$ , $e \in E$ .

It is straightforward to show that $\pi_C : C\natural E \longrightarrow C,\ c\natural e \longmapsto c\,\varepsilon_E(e)$ and $\pi_E : C\natural E \longrightarrow E,\ c\natural e \longmapsto \varepsilon_C(c)e$ are coalgebra surjections since $C$ is a left $H$-comodule coalgebra and $E$ is a left $H$-module coalgebra.

Definition 3 [5]. *Let $H$ be a bialgebra over a field $k$. Let $B$ be a left $H$-module algebra and a left $H$-comodule coalgebra. Let $D$ be a left $H$-comodule algebra and a left $H$-module coalgebra. The generalized biproduct* $B \times_H^L D$ of $B$ and $D$ is defined to be $B\#_H^L D$ as an algebra and $B\natural_H^L D$ as a coalgebra.

EXAMPLE 4. A bialgebra $H$ is a left H-comodule algebra via $\Delta_H$ because $\Delta_H$ is an algebra map. $H$ is a left H-module coalgebra via $m_H$ because $m_H$ is a coalgebra map. The generalized biproduct $B \times_H^L H$ is a biproduct $B \star H$ in [3]. We consider the case when $H = kG$, for $G$ an abelian group. Then $B \times_H^L H = B \star H$ is a bialgebra. As an algebra $B \times_H^L H = B\#H = B * G$, the skew group ring.

DEFINITION 5. *Let $H$ be a bialgebra. Suppose that $B$ is a left $H$-module algebra and a left $H$-comodule coalgebra and $D$ is a left $H$-comodule algebra and a left $H$-module coalgebra. In case $(B\times_H^L D,\ m_{B\#_H^L D},\ \eta_{B\#_H^L D},\ \Delta_{B\sharp_H^L D},$ $\varepsilon_{B\sharp_H^L D})$ is a bialgebra, we say the pair $(D, B)$ is admissible.*

Throughout we let $H$ be a bialgebra over $k$. Suppose $B$ is a left $H$-module algebra and a left $H$-comodule coalgebra and $D$ is a left $H$-comodule algebra and a left $H$-module coalgebra.

DEFINITION 6. *Let $(D, B)$ be an admissible pair and suppose that $A$ be a bialgebra over $k$. Then*

$$B \leftrightarrows_{j}^{\Pi} A \rightleftarrows_{i}^{\pi} D$$

*is an admissible mapping system* if the following conditions hold :

(a) $\Pi \circ j = I_B, \quad \pi \circ i = I_D$,

(b) $i$ and $\pi$ are algebra maps and coalgebra maps, $j$ is an algebra map, and $\Pi$ is a coalgebra map,

(c) $\Pi$ is a $D$-bimodule map ($A$ is given the $D$-bimodule structure via pullback along $i$ and $B$ is given the trivial right $D$-module structure),

(d) $j(B)$ is a sub-$D$-bicomodule of $A$ and $\Pi|_{j(B)}$ is a $D$-bicomodule map ($A$ is given the $D$-bicomodule structure via pushout along $\pi$, $B$ is given the trivial right $D$-comodule structure), and

(e) $(j \circ \Pi) * (i \circ \pi) = I_A$.

PROPOSITION 7 [5]. *Let $(D, B)$ be an admissible pair. Then*

$$B \leftrightarrows_{j_B}^{\Pi_B} B \times_H^L D \rightleftarrows_{i_D}^{\pi_D} D$$

*is an admissible mapping system where $i_D : D \longrightarrow B \times_H^L D, d \mapsto 1_B \times_H^L d$, $\quad j_B : B \longrightarrow B\times_H^L D, b \mapsto b\times_H^L 1_D$, $\quad \Pi_B : B\times_H^L D \longrightarrow B, b\times_H^L d \mapsto \varepsilon_D(d)b$*

and $\pi_D : B \times^L_H D \longrightarrow D, b \times^L_H d \mapsto \varepsilon_B(b)d$.

Next result gives two mapping description of $B \underset{j_B}{\overset{\Pi_B}{\leftrightarrows}} B \times^L_H D \underset{i_D}{\overset{\pi_D}{\rightleftarrows}} D$.

Proposition 8 [5]. *Let $(D, B)$ be an admissible pair and let $A$ be a bialgebra over $k$. Suppose that $B \underset{j}{\overset{\Pi}{\leftrightarrows}} A \underset{i}{\overset{\pi}{\rightleftarrows}} D$ is an admissible mapping system.*

*(1) There exists a unique algebra map $f : B \times^L_H D \longrightarrow A$ such that the diagram*

$$\begin{tikzcd}
 & B \times^L_H D \arrow[dd, "f"] & \\
B \arrow[ur, "j_B"] \arrow[dr, "j"'] & & D \arrow[ul, "i_D"'] \arrow[dl, "i"] \\
 & A &
\end{tikzcd}$$

*commutes. Furthermore the diagram*

$$\begin{tikzcd}
 & B \times^L_H D \arrow[dl, "\Pi_B"'] \arrow[dr, "\pi_D"] \arrow[dd, "f"] & \\
B & & D \\
 & A \arrow[ul, "\Pi"] \arrow[ur, "\pi"'] &
\end{tikzcd}$$

*commutes and $f$ is a bialgebra isomorphism.*

*(2) There exists a unique coalgebra map $g : A \longrightarrow B \times^L_H D$ such that the diagram*

$$\begin{tikzcd}
& B \times^L_H D \arrow[dl, "\Pi_B"'] \arrow[dr, "\pi_D"] & \\
B & & D \\
& A \arrow[ul, "\Pi"] \arrow[ur, "\pi"'] \arrow[uu, "g"] &
\end{tikzcd}$$

*commutes. Furthermore the diagram*

$$\begin{tikzcd}
& B \times^L_H D & \\
B \arrow[ur, "j_B"] \arrow[dr, "j"'] & & D \arrow[ul, "i_D"'] \arrow[dl, "i"] \\
& A \arrow[uu, "g"] &
\end{tikzcd}$$

*commutes and $g$ is a bialgebra isomorphism.*

Let $(D, B)$ be an admissible pair. Then recall that $B \times^L_H D \rightleftarrows^{\pi_D}_{i_D} D$ are bialgebra maps satisfying $\pi_D \circ i_D = I_D$ by Proposition 7. We will solve a converse in case $D$ is a Hopf algebra.

Theorem 9. *Let $D$ be a Hopf algebra with antipode $s_D$ and be a left $H$-comodule algebra and a left $H$-module coalgebra over a field $k$. Let $A$ be a bialgebra over $k$. Suppose $A \rightleftarrows^{\pi}_{i} D$ are bialgebra maps satisfying $\pi \circ i = I_D$. Set $\Pi = I_A * (i \circ s_D \circ \pi)$ and let $B = \Pi(A)$. Let $j : B \to A$ be the inclusion. Then*

*(i) $B$ is a subalgebra of $A$ and $B$ has a coalgebra structure such that $\Pi$ is a coalgebra map.*

*(ii) $B$ is a left $D$-comodule coalgebra, a left $D$-module algebra, a left $D$-comodule algebra and $B$ is a left $D$-module coalgebra.*

*Suppose that $\Pi$ is an algebra map. Then*

*(iii) $B$ is a left $H$-module algebra and $B$ is a left $H$-comodule coalgebra,*

*(iv) $(D,B)$ is an admissible pair and $B \leftrightarrows_{j}^{\Pi} A \rightleftarrows_{i}^{\pi} D$ is an admissible mapping system,*

*(v) The map $f : B \times_H^L D \longrightarrow A, \quad b \times d \mapsto bi(d)$ is an isomorphism of bialgebras.*

*Proof.* In the convolution algebra $End_k(A)$, for all $a \in A$,

$$\begin{aligned}(i \circ s_D \circ \pi) * (i \circ \pi)(a) &= \Sigma(i \circ s_D \circ \pi)(a_1)(i \circ \pi)(a_2)\\ &= \Sigma i(s_D(\pi(a_1)))i(\pi(a_2)) = \Sigma i(s_D(\pi(a_1))\pi(a_2))\\ &= \Sigma i(s_D(\pi(a)_1)\pi(a)_2) = i(\varepsilon_D(\pi(a))1_D) = \varepsilon_D(\pi(a))1_A\\ &= \varepsilon_A(a)1_A = u_A\varepsilon_A(a).\end{aligned}$$

Therefore we have $i \circ s_D \circ \pi = (i \circ \pi)^{-1}$.

Hence

$$\begin{aligned}(j \circ \Pi) * (i \circ \pi) &= [j \circ (I_A * (i \circ s_D \circ \pi))] * (i \circ \pi)\\ &= [j \circ (I_A * (i \circ \pi)^{-1})] * (i \circ \pi) = I_A.\end{aligned}$$

Then we have $(j \circ \Pi) * (i \circ \pi) = I_A$ $\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$ (1)

For all $a, a' \in A$,

$$\begin{aligned}\Pi(aa') &= (I_A * (i \circ s_D \circ \pi))(aa')\\ &= (I_A * (i \circ s_D \circ \pi))(\Sigma a_1a'_1 \otimes a_2a'_2)\\ &= \Sigma I_A(a_1a'_1)(i \circ s_D \circ \pi)(a_2a'_2)\\ &= \Sigma a_1a'_1(i \circ s_D)(\pi(a_2)\pi(a'_2))\\ &= \Sigma a_1a'_1(i \circ s_D \circ \pi)(a'_2)(i \circ s_D \circ \pi)(a_2)\\ &= a_1\Pi(a')(i \circ s_D \circ \pi)(a_2).\end{aligned}$$

Hence we have $\Pi(aa') = a_1\Pi(a')(i \circ s_D \circ \pi)(a_2)$ $\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$ (2)

and

$$\begin{aligned}\Delta(\Pi(a)) &= \Delta(I_A * (i \circ s_D \circ \pi))(a)\\ &= \Delta(\Sigma a_1(i \circ s_D \circ \pi)(a_2))\\ &= \Sigma a_{11}[(i \circ s_D \circ \pi)(a_2)]_1 \otimes a_{12}[(i \circ s_D \circ \pi)(a_2)]_2\\ &= \Sigma a_{11}[i((s_D \circ \pi)(a_2))]_1 \otimes a_{12}[i((s_D \circ \pi)(a_2))]_2\\ &= \Sigma a_{11}[i(((s_D \circ \pi)(a_2))_1)] \otimes a_{12}[i(((s_D \circ \pi)(a_2))_2)]\\ &= \Sigma a_{11}[i(s_D(\pi(a_2))_1)] \otimes a_{12}[i(s_D(\pi(a_2))_2)]\\ &= \Sigma a_{11}[i(s_D(\pi(a_2)_2)] \otimes a_{12}[i(s_D(\pi(a_2)_1)]\\ &= \Sigma a_{11}i(s_D(\pi(a_{22}))) \otimes a_{12}i(s_D(\pi(a_{21})))\end{aligned}$$

$$= \Sigma a_1(i \circ s_D \circ \pi)(a_4) \otimes a_2(i \circ s_D \circ \pi)(a_3)$$
$$= \Sigma a_1(i \circ s_D \circ \pi)(a_3) \otimes \Pi(a_2)$$

Therefore we have $\Delta(\Pi(a)) = \Sigma a_1(i \circ s_D \circ \pi)(a_3) \otimes \Pi(a_2)$. $\cdots\cdots\cdots\cdots$ (3)

For all $d \in D$,

$$\begin{aligned}\Pi(i(d)) &= (I_A * (i \circ s_D \circ \pi))(i(d)) = \Sigma i(d)_1(i \circ s_D \circ \pi)(i(d)_2)\\ &= \Sigma i(d_1)(i \circ s_D \circ \pi)(i(d)_2) = \Sigma i(d_1)i(s_D((\pi \circ i)(d_2)))\\ &= \Sigma i(d_1 s_D((\pi \circ i)(d_2)) = \Sigma i(d_1 s_D(d_2)))\\ &= i(\Sigma d_1 s_D(d_2)) = i(\varepsilon(d)1_D) = \varepsilon(d)1_A.\end{aligned}$$

Therefore we have $\Pi(i(d)) = \varepsilon(d)1_A$. $\cdots\cdots\cdots\cdots\cdots\cdots$ (4)

For all $a \in A$,

$$\begin{aligned}(\pi \circ \Pi)(a) &= \pi \circ (I_A * (i \circ s_D \circ \pi))(a) = \pi(\Sigma a_1(i \circ s_D \circ \pi)(a_2))\\ &= \Sigma \pi(a_1)((\pi \circ i)((s_D \circ \pi)(a_2))) = \Sigma \pi(a_1)(s_D \circ \pi)(a_2)\\ &= \Sigma \pi(a)_1 s_D(\pi(a_2)) = \Sigma \pi(a)_1 s_D(\pi(a)_2)\\ &= \varepsilon_D(\pi(a))1_D = \varepsilon_A(a)1_D.\end{aligned}$$

Therefore we have $(\pi \circ \Pi)(a) = \varepsilon_A(a)1_D$. $\cdots\cdots\cdots\cdots\cdots\cdots$ (5)

By (2) and (4),

$$\begin{aligned}\Pi(ai(d)) &= \Sigma a_1 \Pi(i(d))(i \circ s_D \circ \pi)(a_2) = \Sigma a_1 \varepsilon(d)(i \circ s_D \circ \pi)(a_2)\\ &= (\Sigma a_1(i \circ s_D \circ \pi)(a_2))\varepsilon(d) = \varepsilon(d)\Pi(a).\end{aligned}$$

Therefore we have $\Pi(ai(d)) = \varepsilon(d)\Pi(a)$. $\cdots\cdots\cdots\cdots\cdots\cdots$ (6)

For all $b \in B = \Pi(A)$,

$$\Delta(b) = \Delta(\Pi(a)) = \Sigma a_1(i \circ s_D \circ \pi)(a_3) \otimes \Pi(a_2).$$

Therefore

$$\begin{aligned}\Sigma b_1 \otimes \pi(b_2) &= \Sigma a_1(i \circ s_D \circ \pi)(a_3) \otimes \pi(\Pi(a_2))\\ &= \Sigma a_1(i \circ s_D \circ \pi)(a_3) \otimes \varepsilon_A(a_2)1_D = \Sigma(i \circ s_D \circ \pi)(\varepsilon_A(a_2)a_3) \otimes 1_D\\ &= \Sigma a_1(i \circ s_D \circ \pi)(a_2) \otimes 1_D = \Pi(a) \otimes 1_D = b \otimes 1.\end{aligned}$$

Therefore we have $\Sigma b_1 \otimes \pi(b_2) = b \otimes 1$. $\cdots\cdots\cdots\cdots\cdots\cdots$ (7)

Let $A \otimes D \longrightarrow A,\ \ a \otimes d \mapsto a \cdot d = ai(d)$ be a right $D$-module structure map of $A$ and $B \otimes D \longrightarrow B,\ \ \Pi(a) \otimes d \mapsto \Pi(a) \cdot d = \varepsilon(d)\Pi(a)$ be a right $D$-module structure map of $A$. Let $\rho_A : A \longrightarrow A \otimes D,\ \ a \mapsto \Sigma a_1 \otimes \pi(a_2)$ be a right $D$-comodule structure map of $A$ and $\rho_B : B \longrightarrow B \otimes D,\ \ b \mapsto b \otimes 1_D$ be a right $D$-comudule structure map of $B$. By (6) and (7)

$\Pi(a \cdot h) = \Pi(ai(h)) = \Pi(a)\varepsilon(h) = \Pi(a) \cdot h,$

$\Sigma(\Pi(b))_0 \otimes (\Pi(b))_1 = \Pi(b) \otimes 1 = \Sigma\Pi(b_0) \otimes b_1.$

Therefore we have $\Pi$ is a right $D$-module map and $B$ is a right $D$-subcomodule and $\Pi|_B$ is a right $D$-comodule map. $\cdots\cdots$ (8)

By (1) and (7), we have

$$b = I_A(b) = (j \circ \Pi) * (i \circ \pi)(b) = \Sigma(j \circ \Pi)(b_1)(i \circ \pi)(b_2)$$
$$= \Sigma j(\Pi(b_1))i(\pi(b_2)) = \Sigma\Pi(b_1)i(\pi(b_2)) = \Pi(b).$$

Therefore we have $\Pi(b) = b$. $b \in B$ $\cdots\cdots$ (9)

Hence

$$\Pi \circ j = I_B \quad \cdots\cdots (10)$$

By (3) and (7),

$$\rho_A(bb') = \Sigma(bb')_1 \otimes \pi((bb')_2) = \Sigma b_1 b'_1 \otimes \pi(b_2 b'_2) = \Sigma b_1 b'_1 \otimes \pi(b_2)\pi(b'_2)$$
$$= (\Sigma b_1 \otimes \pi(b_2))(\Sigma b'_1 \otimes \pi(b'_2)) = (b \otimes 1)(b' \otimes 1) = bb' \otimes 1.$$

Therefore we have $B$ is a subalgebra with $\Delta(B) \subseteq A \otimes B$ $\cdots\cdots$ (11)

and

the inclusion map $j : B \longrightarrow A$ is an algebra map $\cdots\cdots$ (12)

Let $d \cdot a = ad_i(d \otimes a) = \Sigma i(d_1)ai(s_D(d_2))$ be the adjoint action of $D$ on $A$. By (2),

$$\Pi(i(d)a') = \Sigma i(d)_1 \Pi(a')(i \circ s_D \circ \pi)(i(d)_2)$$
$$= \Sigma i(d_1)\Pi(a')(i \circ s_D)(\pi(i(d_2)) = \Sigma i(d_1)\Pi(a')(i \circ s_D)(d_2)$$
$$= \Sigma i(d_1)\Pi(a')i(s_D(d_2)) = d \cdot \Pi(a').$$

So $d \cdot \Pi(a') = \Pi(i(d)a') \in B$.

Therefore we have $B$ is a left $D$-module under $ad_i$ and $\Pi$ is a left $D$-module map. $\cdots\cdots$ (13)

Define the comultipication on $B$ as $\Delta_B : B \longrightarrow B \otimes B$ by

$$\Delta_B(\Pi(a)) = \Sigma\, \Pi(a_1) \otimes \Pi(a_2).$$

Let $D^+ = D \cap ker(\varepsilon_D)$. By (6), $Ai\ (D^+) \subseteq ker(\Pi)$ since $\Pi(ai(d)) = \varepsilon_D(d)\Pi(a) = 0$. If $\Pi(a) = 0$, then

$a = I(a) = \Sigma\, \Pi(a_1)(i \circ \pi)(a_2) = \Sigma\, \Pi(a_1)(i \circ \pi)(a_2) - \Pi(a) = \Sigma\, \Pi(a_1)(i \circ \pi)(a_2) - \Pi(\Sigma a_1 \varepsilon_A(a_2)) = \Sigma\, \Pi(a_1)[i(\pi(a_2)) - \varepsilon_A(a_2)1_A] = \Sigma\, \Pi(a_1)[i(\pi(a_2) -$

$\varepsilon(a_2)1_A)] \in A\ i(D^+)$ since $\varepsilon_D(\pi(a_2) - \varepsilon(a_2)1_A) = \varepsilon_A(a_2)1_A - \varepsilon_A(a_2)1_A = 0$. Therefore $ker(\Pi) = A\ i(D^+)$. So $ker(\Pi)$ is a coideal of $A$ and $\Delta_B$ is well-defined. Since $\varepsilon_B \circ \Pi = \varepsilon_A$,

$B$ is a coalgebra and $\Pi : A \longrightarrow B$ is a coalgebra map.$\cdots\cdots\cdots$(14)

Since $\Pi$ is a coalgebra map,

$$\pi(b) = \pi(\Pi(a)) = \varepsilon_A(a)1_D = \varepsilon_B(\Pi(a))1_D = \varepsilon_B(b)1_D$$

by (5). So

$$\begin{aligned}\Sigma\pi(b_1) \otimes b_2 = \Sigma\pi(b_1)s_D(1) \otimes \varepsilon(b_3)b_2 &= \Sigma\pi(b_1)s_D(\varepsilon(b_3)1_D) \otimes b_2 \\ = \Sigma\pi(b_1)s_D(\pi(b_3)) \otimes b_2 &= \Sigma\pi(b_1)(s_D \circ \pi)(b_3) \otimes b_2.\end{aligned}$$

Therefore we have $\Sigma\pi(b_1) \otimes b_2 = \Sigma\pi(b_1)(s_D \circ \pi)(b_3) \otimes b_2$.$\cdots\cdots\cdots\cdots$(15)

If we define the left $D$-comodule structure map of $B$ as $\rho'_B(b) = \Sigma\pi(b_1) \otimes b_2$ then $\rho'_B$ is well-defined since $\Delta B \subseteq A \otimes B$. Then

$B$ is a left $D$-comodule under $\rho'_B$ $\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$(16)

By (8),

$(\rho'_A \circ \Pi|_B)(b) = \rho'_A(\Pi(b)) = \Sigma\pi(b_1) \otimes b_2 = \Sigma\pi(b_1) \otimes \Pi(b_2) = (I_D \otimes \Pi|_B)\rho'_B(b)$.

Therefore we have $\Pi|_B$ is a left $D$-comodule map.$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$(17)

Since $A$ is a left $D$-module algebra under $ad_i$ and $B$ is a submodule of $A$,

$B$ is a left $D$-module algebra$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$(18)

For all $d \in D$ and $a \in A$,

$\Delta_D(d \cdot \Pi(a)) = \Delta_D(\Pi(i(d)a)) = \Sigma\Pi(i(d_1)a_1) \otimes \Pi(i(d_2)a_2) = \Sigma d_1 \cdot \Pi(a_1) \otimes d_2 \cdot \Pi(a_2)$

and

$$\begin{aligned}&\varepsilon(d \cdot a) = \varepsilon(\Sigma i(d_1)ai(s_D(d_2))) = \Sigma\varepsilon(i(d_1))\varepsilon(a)\varepsilon(i(s_D(d_2))) \\ &= \Sigma\varepsilon(a)\varepsilon(i(d_1)i(d_2))) = \Sigma\varepsilon(a)\varepsilon(i(d_1 s_D(d_2)) = \varepsilon(a)\varepsilon(i(\varepsilon(d)1)) = \varepsilon(a)\varepsilon(d)1.\end{aligned}$$

Therefore

$B$ is a left $D$-module coalgebra$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$(19)

Since $\rho'_B(b) = \Sigma\pi(b_1) \otimes b_2$ it follows that

$B$ is a left $D$-comodule algebra$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$(20)

$(I \otimes \Pi) \circ \rho'_A(a) = (I \otimes \Pi)(\Sigma\pi(a_1) \otimes a_2) = \Sigma\pi(a_1) \otimes \Pi(a_2) = \Sigma\pi(a_1)s_D \circ$

$\pi(a_3) \otimes \Pi(a_2) = \Sigma\pi(a_1)(\pi \circ i \circ s_D \circ \pi)(a_3) \otimes \Pi(a_2) = \Sigma\pi(a_1(i \circ s_D \circ \pi)(a_3)) \otimes \Pi(a_2) = (\pi \otimes I)(\Sigma a_1(i \circ s_D \circ \pi)(a_3)) \otimes \Pi(a_2)) = (\pi \otimes I)\Delta(\Pi(a)) = \rho'_B(\Pi(a))$. Thus $\Pi : A \longrightarrow B$ is a surjective coalgebra map such that $(I \otimes \Pi) \circ \rho'_A = \rho'_B \circ \Pi$ where $\rho'_B = \rho'_A|_B$. Since $A$ is a left $D$-comodule under $\rho'_A$,

$B$ is a left $D$-comodule coalgebra under $\rho'_B \cdots\cdots\cdots\cdots\cdots\cdots$ (21)

(i) : From (11) and (14).

(ii) : From (18),(19),(20) and (21).

(iii) : We have defined the multiplication and the unit of $B$ as

$m_B : B \otimes B \longrightarrow B,\ \ \Pi(a) \otimes \Pi(a') \mapsto \Pi(a)\Pi(a') = \Pi(aa')$

and

$u_B : k \longrightarrow B,\ \ 1_k \mapsto u_B(1_k) = 1_B = 1_A.$

We have defined the comultipication and the counit of $B$ as

$\Delta_B : B \longrightarrow B \otimes B,\ \ \Pi(a) \mapsto \Delta_B(\Pi(a)) = \Sigma\Pi(a_1) \otimes \Pi(a_2)$

and

$\varepsilon_B : B \longrightarrow k,\ \ \Pi(a) \mapsto \varepsilon_B(\Pi(a)) = \varepsilon_A(a).$

Then $(B, m_B, u_B)$ is a algebra and $(B, \Delta_B, \varepsilon_B)$ is a coalgebra. We define $H \otimes B \to B,\ \ h \otimes \Pi(a) \mapsto h \cdot \Pi(a) = \varepsilon_H(h)\varepsilon_A(a)1_B = \varepsilon_H(h)\varepsilon_B(b)1_B$

and

$\rho''_B : B \longrightarrow H \otimes B,\ \ \Pi(a) \mapsto \varepsilon_A(a)(1_H \otimes (\Pi \circ i)(1_D)) = \varepsilon_A(a)(1_H \otimes 1_B).$

Then $B = \Pi(A)$ is a left $H$-module and $B$ is a left $H$-comodule. For all $\Pi(a) \in B$, we compute

$m_B(h \cdot (\Pi(a) \otimes \Pi(a')) = h \cdot m_B(\Pi(a) \otimes \Pi(a')$

and

$u_B(h \cdot 1_k) = h \cdot u(1_k),$

since $\Pi$ is an algebra map.

Therefore we have $B$ is a left $H$-module algebra.

For all $\Pi(a) \in B$, we compute

$(\rho_{B \otimes B} \circ \Delta_B)(\Pi(a)) = (I \otimes \Delta_B)\rho''_B(\Pi(a))$

and

$((I \otimes \varepsilon) \circ \rho''_B)(\Pi(a)) = (\rho_k \circ \varepsilon_B)(\Pi(a)).$

Therefore we have $B$ is a left $H$-comodule coalgebra.

(iv) : We will show that $(D, B)$ is a admissible pair. Let

$$(b \times d)(b' \times d') = \Sigma b(d_{-1} \cdot b') \times d_0 d' = \Sigma b \varepsilon_H(d_{-1}) \varepsilon_A(a') \times d_0 d'$$
$$= \Sigma \varepsilon_A(a') b \times dd',$$

where $b' = \Pi(a')$.

Then $B \#_H^L D$ is an associative algebra with identity $1_B \# 1_D$ by [6, Proposition 1]. Let

$$\Delta(b \times d) = \Sigma(b_1 \times b_{2,-1} \cdot d_1) \otimes (b_{2,0} \times d_2), \ \ \varepsilon(b \times d) = \varepsilon_B(b)\varepsilon_D(d).$$

Then $B \natural_H^L D$ is a coassociative coalgebra by [6, Proposition 2].

Define $\rho''_A : \longrightarrow H \otimes A, \ \ a \mapsto \varepsilon_A(a)(1_H \otimes 1_A)$. Then $A$ is a left $H$-comodule and $\Pi$ is a left $H$-comodule map. Since $\Pi$ is a coalgebra map and $\Pi$ is a left $H$-comodule map,

$$
\begin{aligned}
&\Delta(b \times d)\Delta(b' \times d') \\
&= [\Sigma(b_1 \times b_{2,-1} \cdot d_1) \otimes (b_{2,0} \times d_2)][(b'_1 \times b'_{2,-1} \cdot d'_1) \otimes (b'_{2,0} \times d'_2)] \\
&= \Sigma(b_1 \times b_{2,-1} \cdot d_1)(b'_1 \times b'_{2,-1} \cdot d'_1) \otimes (b_{2,0} \times d_2)(b'_{2,0} \times d'_2) \\
&= \Sigma[\varepsilon_A(a'_1) b_1 \times (b_{2,-1} \cdot d_1)(b'_{2,-1} \cdot d'_1)] \otimes [\varepsilon_A(a'_{2,0}) b_{2,0} \times d_2 d'_2] \\
&= \Sigma[\varepsilon_A(a'_1)\Pi(a_1) \times (a_{2,-1} \cdot d_1)(a'_{2,-1} \cdot d'_1)] \otimes [\varepsilon_A(a'_{2,0})\Pi(a_{2,0}) \times d_2 d'_2] \\
&= \Sigma[\varepsilon_A(a'_1)\Pi(a_1) \times (\varepsilon_A(a_2)1_H \cdot d_1)(\varepsilon_A(a'_2)1_H \cdot d'_1)] \otimes [\varepsilon_A(1_A)\Pi(1_A) \times d_2 d'_2] \\
&= \Sigma \varepsilon_A(a')(\Pi(a) \times d_1 d'_1) \otimes (1_B \times d_2 d'_2) \\
&= \Sigma \varepsilon_A(a')[\Pi(a_1) \times \varepsilon_A(a_2)1_H \cdot (d_1 d'_1)] \otimes (\Pi(1_A) \times d_2 d'_2) \\
&= \Sigma \varepsilon_A(a')(\Pi(a_1) \times a_{2,-1} \cdot (d_1 d'_1)) \otimes (\Pi(a_{2,0} \times d_2 d'_2) \\
&= \Sigma \varepsilon_A(a')(\Sigma b_1 \times b_{2,-1} \cdot (dd')_1) \otimes (b_{2,0} \times (dd')_2) \\
&= \Sigma \varepsilon_A(a')\Delta(b \times dd') \\
&= \Delta(\Sigma \varepsilon_A(a') b \times dd') \\
&= \Delta((d \times d)(b' \times d'))
\end{aligned}
$$

where $b = \Pi(a)$ and $b' = \Pi(a')$. We have

$$
\begin{aligned}
\Delta(1_B \times 1_D) &= \Sigma[(1_B)_1 \times (1_B)_{2,-1} \cdot (1_D)_1] \otimes [(1_B)_{2,0} \times (1_D)_2] \\
&= \Sigma[\Pi(1_A) \times \Pi(1_A)_{-1} \cdot 1_D] \otimes [\Pi(1_A)_0 \times 1_D] \\
&= \Sigma[1_B \times (1_B)_{-1} \cdot 1_D] \otimes [(1_B)_0 \times 1_D] \\
&= (1_B \times 1_H \cdot 1_D) \otimes (1_B \times 1_D) \\
&= (1_B \times 1_D) \otimes (1_B \times 1_D)
\end{aligned}
$$

by (22).

Therefore we have $\Delta(b\times d)\Delta(b'\times d') = \Delta((b\times d)(b'\times d'))$ and $\Delta(1_B\times 1_D) = (1_B\times 1_d)\otimes(1_B\times 1_D)$. So $\Delta$ is an algebra map. Since $A$ and $D$ are bialgebras, we compute

$$\varepsilon((b\times d)(b'\times d')) = \varepsilon(b\times d)\varepsilon(b'\times d'),\ \ \varepsilon(1_B\times 1_D) = 1_k.$$

So $\varepsilon$ is an algebra map. Therefore we have $B\times_H^L D$ is a bialgebra so $(D,B)$ is an admissible pair. By (1),(12),(13),(16) and (17), $B \leftrightarrows_j^\Pi A \rightleftarrows_i^\pi D$ is an admissible mapping system.

(v) : From (iv) and Proposition 8. □

Remark 10. *If we assume that* $h\cdot 1_D = \varepsilon_H(h)1_D$ *then the* $H$*-module structure of* $B$ *in the proof of Theorem 1 is reduced from the* $H$*-module structure of* $D$ *:*

$$\begin{aligned} h\cdot\Pi(a) &= (\Pi\circ i)(h\cdot\pi(\Pi(a))) = (\Pi\circ i)(h\cdot\varepsilon_A(a)1_D)\\ &= \varepsilon_A(a)(\Pi\circ i)(h\cdot 1_D) = \varepsilon_A(a)(\Pi\circ i)(\varepsilon_H(h)1_D)\\ &= \varepsilon_H(h)\varepsilon_A(a)\Pi(1_A) = \varepsilon_H(h)\varepsilon_A(a)\Pi(1_A) = \varepsilon_H(h)\varepsilon_A(a)1_B\\ &= \varepsilon_H(h)\varepsilon_B(\Pi(a))1_B = \varepsilon_H(h)\varepsilon_B(b)_B. \end{aligned}$$

Corollary 11. *Let* $B$ *be as in the Theorem above. Then the following are equivalent:*

*(1)* $\Pi$ *is an algebra map.*

*(2)* $d\cdot b = \varepsilon(d)b,\ \ d\in D$ *and* $b\in B$.

*Proof.* (1) $\Rightarrow$ (2) : By (4), for all $d\in D$ and $b\in B$,

$$\begin{aligned} d\cdot b &= ad_i(d\otimes a) = \Sigma i(d_1)bi(s_D(d_2)) = \Sigma i(d_1)\Pi(b)(i\circ s_D)(d_2)\\ &= \Sigma i(d_1)\Pi(b)(i\circ s_D\circ\pi)(i(d_2)) = \Sigma i(d)_1\Pi(b)(i\circ s_D\circ\pi)(i(d_2))\\ &= \Pi(i(d)b) = \Pi(i(d))\Pi(b) = \varepsilon(d)b, \end{aligned}$$

since $\Pi$ is an algebra map, $\Pi(b) = b$ and $\pi\circ i = I$.

(2) $\Rightarrow$ (1) : By Theorem 1 (v), $a = bi(d)$. For $a'\in A$,

$$\begin{aligned} \Pi(aa') &= \Pi(bi(d)a') = \Sigma b_1\Pi(i(d)a')(i\circ s\circ\pi)(b_2)\\ &= \Sigma b_1\Pi(i(d)a')(i\circ s)(\pi(b_2)) = b\Pi(i(d)a')(i\circ s)(i_D)\\ &= b\Pi(i(d)a') = b(d\cdot\Pi(a')) = b(\varepsilon(d)\Pi(a')) = \varepsilon(d)b\Pi(a')\\ &= \Pi(bi(d))\Pi(a') = \Pi(a)\Pi(a'), \end{aligned}$$

by the right $D$-module structure of $B$. Therefore $\Pi$ is an algebra map. □

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